# Math 261B Thurs. 10/8

$SO_{2n}$ preserves $(x, y) = x^T J y$ $\quad J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$

Eq'ns $\quad A^R A = I \quad \det A = 1$ $\quad \left( \searrow_R \right)$

$T = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix}$ $\quad X(T) = \mathbb{Z}^n$

Lie Alg : $\quad M^R + M = 0$ $\quad \begin{pmatrix} & & x & 0 \\ & & & -x \\ & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \end{pmatrix}$

Root Spaces $\quad \begin{pmatrix} t_i & x & & y & \\ & t_j & & & -y \\ & & & & -x \\ & & & t_j^{-1} & \\ & & & & t_i^{-1} \end{pmatrix}$

$x \to t_i/t_j \, x$ $\quad$ Root $e_i - e_j$

$y \to t_i t_j \, y$ $\quad$ $e_i + e_j$

Roots $\pm e_i \pm e_j$

Root $SL_2$'s $\quad \begin{pmatrix} a & b & & \\ c & d & & \\ & & a & -b \\ & & -c & d \end{pmatrix}$

$t_i \mapsto t \quad t_j \mapsto t^{-1}$ $\quad \underset{\alpha^\vee}{\varepsilon_i - \varepsilon_j} \leftrightarrow \underset{\alpha}{e_i - e_j}$

or $\quad t_i \to t \quad t_j \to t$ $\quad \underset{\alpha^\vee}{\varepsilon_i + \varepsilon_j} \leftrightarrow \underset{\alpha}{e_i + e_j}$

$\langle \alpha_{n-1}^\vee, \alpha_n \rangle$

$R_+$ $\quad e_i \pm e_j \quad i<j \quad \alpha_1 \qquad \alpha_n$

Simple roots: $\quad e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_{n-1} - e_n,\ \ e_{n-1} + e_n$

$e_i + e_j \qquad\qquad e_i + e_n = e_i - e_{i+1} + \cdots + (e_{n-2} - e_{n-1}) + e_{n-1} + e_n$

$\quad \| $

$e_i + e_n + (e_{n-1} - e_n) + \cdots + (e_j - e_{j+1})$

$\alpha = e_i - e_{i+1} \quad \alpha^\vee = \varepsilon_i - \varepsilon_{i+1} \longrightarrow s_i \quad e_i \leftrightarrow e_{i+1}$

$\alpha = e_{n-1} + e_n \quad \alpha^\vee = \varepsilon_{n-1} + \varepsilon_n \longrightarrow s_n \quad e_{n-1} \leftrightarrow -e_n \qquad (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, -x_n, -x_{n-1})$

$W$ = subgroup of $B_n$ with even # of sign changes

$= D_n$

$\langle \varepsilon_i - \varepsilon_{i+1},\ e_{n-1} + e_n \rangle$

Cartan matrix $\langle \alpha_j^\vee, \alpha_i \rangle$

$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & 0 & \\ & -1 & 2 & & & & \\ & & & \ddots & & & \\ & & & & 2 & -1 & -1 \\ & 0 & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}$$

Dynkin diagram


$D_n$

$Q = \{\lambda \mid \sum x_i \text{ even}\} \subseteq X = \mathbb{Z}^n$ $\quad X/Q = \mathbb{Z}/2\mathbb{Z}$ $\quad |Z(SO_{2n})| = 2$

$Q^\vee = \{\beta \mid \sum \beta_i \text{ even}\} \subseteq X^* = \mathbb{Z}^n$ $\quad\quad Z(SO_{2n}) = \{\pm I\}$

index 2 in $X^*$ $\quad$ Adjoint group is $SO_{2n}/\{\pm I\}$

Another "$X$" is $X^* = Q^\vee$, $X = (Q^\vee)^*$ $\quad$ (replace $X$ with $Q$, $X^*$ with $Q^*$)

$$(Q^\vee)^* = \mathbb{Z}^n \amalg \left(\mathbb{Z}^n + \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right)\right)$$

Gives simply connected covering group $Spin_{2n} \to SO_{2n}$

$\lambda \in X$ is <u>dominant</u> if $\langle \alpha_i^\vee, \lambda \rangle \geq 0$ $\quad$ (for $GL_n$: $\lambda_1 \geq \dots \geq \lambda_n$)

$\lambda_1 \geq \dots \geq \lambda_n, \quad \lambda_n + \lambda_{n-1} \geq 0$

$(1, \dots, 1, 0, \dots, 0) \leftarrow$ H.W of $\Lambda^k V$

($SO_{2n+1}$: $\lambda_1 \geq \dots \geq \lambda_n \geq 0$)

Defining rep $V = K^N$ has weights $\pm e_i$ $(N = 2n)$ $\quad \pm e_i, 0$ $(N = 2n+1)$

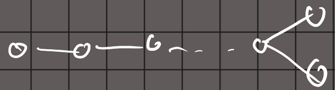
highest: $e_1$ $\quad\quad e_1$

$Spin_{2n+1}$ : $\langle \varepsilon_i - \varepsilon_{i+1}, \lambda \rangle = 0$ $\quad \langle 2\varepsilon_n, \lambda \rangle = 1$ $\quad \lambda = \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right)$

$\quad\quad$ HW of a $2^n$ dim'l irrep of $Spin_{2n+1}$: <u>Clifford module</u>

$Spin_{2n}$ : $\quad \langle \varepsilon_{n-1} + \varepsilon_n, \lambda \rangle = 1$

$\lambda = \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right)$



$\lambda = (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$

$\langle \varepsilon_{n+1} + \varepsilon_n, \lambda \rangle = 0$

$\langle \varepsilon_{n+1} - \varepsilon_n, \lambda \rangle = 1$

$SO_{2n}^L = SO_{2n}$ $\quad$ $Spin_{2n}^L = SO_{2n}/\{\pm I\}$ $\quad$ $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$

$SO_2 = \mathbb{G}_m$ no roots!

$SO_3 \cong PGL_2$

$SO_4$ $\quad \circ \quad \circ$

$$\begin{pmatrix} a & b & & \\ c & d & & \\ & & a & -b \\ & & -c & d \end{pmatrix}, \begin{pmatrix} a & & b & \\ & a & & -b \\ c & & d & \\ & -c & & d \end{pmatrix}$$

$SL_2 \times SL_2 \to SO_4 = SL_2 \times SL_2 / (-I_2, -I_2)$

$$Spin_4 \cong SL_2 \times SL_2 \to SL_2 \times PSL_2 \to SO_4/\{\pm 1\}$$
$$Spin_4 \cong SL_2 \times SL_2 \to SO_4 \to SO_4/\{\pm 1\}$$
$$Spin_4 \cong SL_2 \times SL_2 \to PSL_2 \times SL_2 \to SO_4/\{\pm 1\}$$
$$SO_4/\{\pm 1\} \cong PSL_2 \times PSL_2$$

$SO_{2n}$ $\quad |X/Q| = 2$ $\quad |X^*/Q^\vee| = 2$

$|(Q^\vee)^*/Q| = 4$ $\quad |Q^\vee/Q^\vee| = 1$

$n$ even: $(\mathbb{Z}/2\mathbb{Z})^2 = Z(Spin_{2n})$

$n$ odd: $\mathbb{Z}/4\mathbb{Z}$

$Spin_{2n} \to SO_{2n} \to SO_{2n}/\{\pm 1\}$.

$$Spin_{2n} \to ? \to SO_{2n}/\{\pm 1\}$$
$$Spin_{2n} \to SO_{2n} \to SO_{2n}/\{\pm 1\}$$
$$Spin_{2n} \to ? \to SO_{2n}/\{\pm 1\}$$

# Symplectic group

$U = K^n \qquad V = U \oplus U^*$

$\langle , \rangle : V \otimes V \to \Lambda^2 V \to K$

$\langle (v,f), (w,g) \rangle = g(v) - f(w)$

$\langle x,y \rangle = -\langle y,x \rangle \iff \langle x,x \rangle = 0$

$x \wedge x = 0 \Rightarrow x \wedge y = -y \wedge x$

$(x+y) \wedge (x+y) = 0$

$\underbrace{x\wedge x}_{0} + \underbrace{x \wedge y + y \wedge x}_{0} + \underbrace{y \wedge y}_{0}$

$u_1, \dots, u_n \in U$ dual $\varepsilon_1, \dots, \varepsilon_n \in U^*$

$e_1, \dots, e_{2n} = (u_1, \dots, u_n, \varepsilon_n, \dots, \varepsilon_1)$

$\Updownarrow$

Matrix of $\langle , \rangle$: $\quad \langle \underline{x}, \underline{y} \rangle = x^T J_- y \qquad J_- = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$

$Sp_{2n} = \{ A \in GL(V) : \langle A\underline{x}, A\underline{y} \rangle = \langle \underline{x}, \underline{y} \rangle \}$

$(SL(V))$

$x^T A^T J_- A y = x^T J_- y$

$A^T J_- A = J_- \qquad J_-^2 = -I$

$(-J_- A^T J_-) A = I$

$I_- = \begin{pmatrix} I & \\ & -I \end{pmatrix}$

$$J_- = I_- J = -J I_- \qquad J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

$$-J_- A^T J_- = I_- J A^T J I_- = I_- A^R I_-$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} D^R & -B^R \\ -C^R & A^R \end{pmatrix}$$

Ex. $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \overset{!}{=} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \iff \det = 1$$

(require for $Sp_2$)

$$Sp_2 = Sl_2$$

Lie algebra: $A = I + \varepsilon M$ $\quad \varepsilon^2 = 0$ $\quad \langle Ax, Ay \rangle = \langle x, y \rangle$

$$\langle Mx, y \rangle + \langle x, My \rangle = 0$$

$$x^T J_- y$$

$$x^T M^T J_- y + x^T J_- M^T y = 0$$

$$M^T J_- + J_- M = 0 \qquad J_-^2 = -I$$

$$-J_- M^T J_- + M = 0$$

$$M = \begin{pmatrix} A & B = B^R \\ C = C^R & -A^R \end{pmatrix}$$

$$T = \begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix}$$

$$X(T) = \mathbb{Z}^n$$

$$\left(\begin{array}{cc|cc} t_i^{\circ} & x & y & z \\ & t_j^{\circ} & & y \\ \hline & & \ddots & -x \\ & & t_j^{-1} & \ddots \\ & & & t_i^{-1} \end{array}\right)$$

| | Root |
|---|---|
| $x \mapsto t_i/t_j \, x$ | $e_i - e_j$ |
| $y \mapsto t_i t_j \, y$ | $e_i + e_j$ |
| $z \mapsto t_i^2 \, z$ | $2e_i$ |

Roots: $\pm e_i \pm e_j, \ \pm 2e_i$

$e_i - e_j \leftrightarrow \varepsilon_i - \varepsilon_j$, $\quad e_i + e_j \leftrightarrow \varepsilon_i + \varepsilon_j$

Coroots $\longrightarrow$ $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$ $\leftarrow R_+, R_+^\vee$ $(i<j)$

$2e_i \leftrightarrow \varepsilon_i$

$$\begin{pmatrix} a & & b & \\ & a & ? & b \\ c & & d & \\ & c & & d \end{pmatrix}$$

$$\begin{pmatrix} a & b & & \\ c & d & & \\ & & a & -b \\ & & -c & d \end{pmatrix}$$

$t$ ; $b,\ t^{-1}$ ; $d^2$

$Q$ = even vectors

$$X/Q = \mathbb{Z}/2\mathbb{Z}$$

$$\begin{pmatrix} t & a & & b \\ & c & & d \\ & & & t^{-1} \end{pmatrix}$$

$t_i \mapsto t$

$$Sp_{2n}^{L} = SO_{2n+1}$$

$$Sp_{2n}/\{\pm I\} = Spin_{2n+1}^{L}$$

Puzzle: char $k = 2$

$SO_{2n}$ as we defined it is $Sp_{2n}$ !!

$SO_{2n+1}$ is non-reduced ...

trouble

e.g. $SO_1$

OK